NEARLY-OPTIMAL ESTIMATES FOR THE STABILITY PROBLEM IN HARDY SPACES

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ABSTRACT. We continue the work of [14]. Let E be a non-Blaschke subset of the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$. Fixed $1 \leq p \leq \infty$, let $H^p(\mathbb{D})$ be the Hardy space of holomorphic functions in the disk whose boundary value function is in $L^p(\partial\mathbb{D})$. Fixed $0 < R < 1$. For $\epsilon > 0$ define

$$C_p(\varepsilon, R) = \sup\{\sup_{|z|\leq R} |g(z)| : g \in H^p,\ \|g\|_p \leq 1,\ |g(\zeta)| \leq \varepsilon\ \forall\zeta \in E\}.$$

In this paper we find upper and lower bounds for $C_p(\epsilon, R)$ when ϵ is small for any non-Blaschke set E. The bounds are nearly-optimal for many such sets E, including sets contained in a compact subset of $\mathbb{D}$ and sets contained in a finite union of Stolz angles.

1. Introduction

This work is a continuation of [14]. The purpose of this paper is to find good estimates for the stability problem of approximating analytic functions in Hardy spaces.

Let E be a subset of the unit disc $\mathbb{D}$ of the complex plane C. To avoid trivial counter-examples, we assume throughout this paper that E is non-Balschke, that is

(B): E contains a non-Blaschke sequence (z_j), that is, a sequence satisfying the condition

$$\sum_{j=1}^{\infty}(1-|z_j|) = \infty.$$

Also without loss of generality, we assume throughout that E is relatively closed in $\mathbb{D}$, that is if $\overline{E}$ is the closure of E in the usual topology in $\mathbb{C}$ then $\overline{E} \cap \mathbb{D} = E$.

Fixed $1 \leq p \leq \infty$, recall that the Hardy space $H^p(\mathbb{D})$ is the space of all holomorphic functions g on $\mathbb{D}$ for which $\|g\|_p < \infty$, where

$$\|g\|_p = \lim_{r\uparrow 1}\left\{\frac{1}{2\pi}\int_0^{2\pi}|g(re^{i\theta})|^p d\theta\right\}^{1/p} \qquad (1 \leq p < \infty),$$
$$\|g\|_\infty = \lim_{r\uparrow 1}\sup_\theta |g(re^{i\theta})|.$$

For convenience, from now on, we will denote $H^p(\mathbb{D})$ by H^p. We define $\mathcal{A}^p$ to be the functions in H^p with norm 1, that is

$$\mathcal{A}^p = \{f :\ f \in H^p, ||f||_{H^p} = 1\}. \tag{1.1}$$

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If $f \in \mathcal{A}^p$ it follows that (see Section 2)

$$|f(z)| \leq \frac{1}{(1-|z|^2)^{1/p}}, \tag{1.2}$$

for all $z \in \mathbb{D}$.

If f is a function in $H^p(\mathbb{D})$ then it is well-known that f can be reconstructed from its values $f(\zeta)$ at points $\zeta \in E$ (see Theorem 7). However, in practice, it is usually the case that we do not know exact values $f(\zeta)$, but only approximate values. This leads to the stability problem, that of estimating the quantity

$$C_p(E,\varepsilon,R) = \sup\{\sup_{|z|\leq R} |g(z)| : g \in \mathcal{A}^p, |g(\zeta)| \leq \varepsilon \ \forall \zeta \in E\}, \tag{1.3}$$

for positive ε and R in $(0,1)$. We can also consider the problem of one-point estimation, which is estimating the number

$$C_p(E,\varepsilon,0) = \sup\{|g(0)| : g \in \mathcal{A}^p, |g(\zeta)| \leq \varepsilon \ \forall \zeta \in E\}. \tag{1.4}$$

Since E satisfies (B), it is well-known that

$$\lim_{\varepsilon \to 0} C_p(\varepsilon, R) = 0.$$

This problem of estimating $C_p(E,\epsilon,R)$ was thoroughly explored by many authors. Let us recall some of the results known in literature.

In [3], Lavrent'ev, Romanov and Shishat-skii used a certain characteristic of the projection of E onto the real axis, to show that if $E \subset U = \{z : |z| \leq 1/4\}$ then $C_p(\varepsilon,R) \leq \max\{\varepsilon^{4/25}, (6/7)^{n(\varepsilon)}\}$ for all $R \in (0,1/4)$, in which $n(\varepsilon) \to \infty$ as $\varepsilon \to 0$. This approach is quite interesting in that E could be a sequence. However in their approach the set E is strictly contained inside $\mathbb{D}$, and only upper bounds are obtained.

In a series of works ([5], [6], [7], [8], [9], [10] and [11]), Osipenko obtained optimal estimates for some special sets E. For example, when E is contained in the real open interval $(-1,1)$ and satisfies some more constrains, he showed that the optimal value of $C_p(E,\epsilon,0)$ is obtained at a finite Balschke product $B(z)$ with all zeros in E, that is

$$B(z) = \prod_{j=1}^{n} \frac{z - z_j}{1 - \overline{z}_j z},$$

and here $z_j \in \overline{E} \subset \mathbb{D}$. It is interesting that here the set E needs not to be contained in a compact set of $(-1,1)$. However, his method seems not applicable to more general sets E.

In case $p = \infty$, it is well-known that the set of boundary limit points, or more exactly non-tangential limit points, of E plays an important role in estimating $C_\infty(E,\epsilon,R)$. Let us first recall the definition of non-tangential limit points E_0 of E (see [2]):

Definition 1. *For each set E of $\mathbb{D}$, we denote by E_0 the set of nontangential limit points of E, that is, points ζ of $\partial\mathbb{D}$ being such that there exists a sequence (z_n) in E which tends nontangentially to ζ, that is, such that*

$$z_n \to \zeta, \ |z_n - \zeta| = O(1 - |z_n|).$$

Let $m(E_0)$ be the Lebesgue measure of E_0 as a subset of $\partial\mathbb{D}$. If $m(E_0) > 0$, we can use the harmonic measure $\omega(z)$ of E_0 to obtain the following estimate (see the Appendix):

$$\epsilon \leq C_p(E,\epsilon,R) \leq \frac{2^{1/p}}{(1-R^2)^{1/p}} \sup_{|z|\leq R} \epsilon^{\omega(z)}. \tag{1.5}$$

Hence in case $m(E_0) > 0$ we obtain a quasi-polynomial estimate for $C_p(E,\epsilon,R)$.

The main purpose of this paper is to obtain good upper and lower bounds for $C_p(E,\epsilon,R)$ for the remaining case when $m(E_0) = 0$ in such a way to extend the above mentioned results of Lavrente's et al. and Osipenko. Our idea consists of two steps:

-Step 1: Use the interpolation by finite Balschke product to reduce estimating $C_p(E,\epsilon,R)$ to estimating of some expressions depending only on ϵ and finite Blaschke products with all zeros in E. This step 1 was already done in our previous paper (see Section 3 in [14]), where an algorithm for choosing the interpolation points was proposed. However that algorithm depends on the ordering of the sequence (z_k), and the method used there does not allow obtaining lower bounds for $C_p(E,\epsilon,R)$. We propose a better algorithm in Step 2 below, which allows us to obtain both upper and lower estimates for $C_p(E,\epsilon,R)$, and to obtain nearly-optimal estimates for many sets E (see Corollaries 1 and 2).

-Step 2: For any $n \geq 0$, assigns a number $M_n(E)$ using finite weighted-Blaschke products (see Definition 3) to construct set functions for E. Then we use these functions $M_n(E)$ to estimate the expressions in Step 1.

Explicitly we fix a bounded holomorphic function $q(z)$ in $\mathbb{D}$ satisfying the following conditions: $q(z) \neq 0$ for all $z \in \mathbb{D}$ and

$$\lim_{z\in E, z\to\partial\mathbb{D}} q(z) = 0. \tag{1.6}$$

The function $q(z)$ mentioned above is provided by the following Theorem by Hayman[2]:

Theorem 2. *If the set E_0 of nontangential limit points of a set E has positive linear measure and if f is a bounded analytic function satisfying*

$$\lim_{z\in E,\ |z|\to 1} f(z) = 0,$$

then $f \equiv 0$. Conversely, if E_0 has measure zero, then there exists $f(z)$, such that $0 < |f(z)| < 1$ in U, and satisfying

$$\lim_{z\in E,\ |z|\to 1} f(z) = 0.$$

Before stating our main results, let us fix some notations.

Definition 3. *We will use the notation $Z_n = \{z_1,\dots,z_n\}$ to denote a tuple of n-points $z_1,\dots,z_n \in \mathbb{D}$. If $j \in \{1,\dots,n\}$ we define $Z_{n,j} = \{z_1,\dots,z_n\}\backslash\{z_j\}$. Define $B(Z_n,z)$ to be the Blaschke product with zeros in Z_n:*

$$B(Z_n,z) = \prod_{j=1}^{n} \frac{z-z_j}{1-\overline{z}_j z}.$$

Similarly define $B(Z_{n,k},z)$ to be the Blaschke product with zeros in $Z_{n,k}$:

$$B(Z_{n,k},z) = \prod_{1\leq j\leq n,\ j\neq k} \frac{z-z_j}{1-\overline{z}_j z}.$$

For a fixed function $q(z)$*, the weighted Blaschke product* $B_q(Z_n, z)$ *is defined as*

$$B_q(Z_n, z) = q(z) \prod_{j=1}^{n} \frac{z - z_j}{1 - \overline{z}_j z}.$$

Let $q(z)$ be a function provided by Theorem 2. Let us define

$$g(E, \epsilon, R, q) = \sup\{ \sup_{|z| \leq R} |B_q(Z_n; z)| : \ n \in \mathbb{N}, \ Z_n \in E^n, \ |B_q(Z_n; \zeta)| \leq \epsilon \ \forall \zeta \in E\}. \tag{1.7}$$

(Note that $g(E, \epsilon, R, q)$ does not depend on p.)

Theorem 4. *Let* $E \subset \mathbb{D}$ *be such that* $m(E_0) = 0$*. Fix* $1 \leq p \leq \infty$ *and* $0 < R < 1$*. Let* $q(z)$ *be a function provided by Theorem 2, normalized by* $||q||_\infty = 1$ *where* $||q||_\infty$ *is its usual sup-norm. Define* $g(E, \epsilon, R, q)$ *as in (1.7). Then there exists* $\epsilon_0 > 0$ *depending on* E *and* $q(z)$*, and a non-increasing function* $\varphi : (0, \epsilon_0) \to (0, \infty)$ *also depending on* E *and* $q(z)$ *satisfying*

$$\lim_{\epsilon \to 0} \varphi(\epsilon) = 0,$$

, a constant $K > 0$ *depending only on* p *and* R*, and a constant* $\alpha > 0$ *depending only on* R*, such that for all* $0 < \epsilon < \epsilon_0$ *we have*

$$g(E, \epsilon, R, q) \leq C_p(E, \epsilon, R) \leq K \times |q(0)|^{-\alpha} \times g^{\alpha}(E, \varphi(\epsilon), R, q). \tag{1.8}$$

A class of sets E satisfying the condition $m(E_0) = 0$ are those contained in a finite union of Stolz angles, which we recall in the following

Definition 5. *Let* $\zeta \in \partial\mathbb{D}$*. A Stolz angle with vertex* ζ *is a set of the form*

$$\Omega_\sigma(\zeta) := \{z \in \mathbb{D} : \ |1 - \overline{z}\zeta| \leq \sigma(1 - |z|)\},$$

where $\sigma \geq 1$ *is some constant.*

The following corollaries can be considered as extensions of above results of Lavrent'ev et al. and Osipenko:

Corollary 1. *If* E *is a compact subset in* $\mathbb{D}$ *then there exist constants* $K > 0$ *and* $\epsilon_0 > 0$ *depending only on* p *and* R*, and there exists a constant* $\alpha > 0$ *depending only on* R *such that for all* $0 < \epsilon < \epsilon_0$*, there exists a finite Blaschke product* $B(z)$ *with all zeros in* E *such that*

$$\sup_{|z| \leq R} |B(z)| \leq C_p(E, \epsilon, R) \leq K \times \sup_{|z| \leq R} |B(z)|^{\alpha}.$$

Corollary 2. *If* E *is contained in a finite union of Stolz angles then there exist constants* K_p*,* $\sigma > 0$ *and* $\epsilon_0 > 0$ *depending only on* R *and the vertices of these Stolz angles, and there exists a constant* $\alpha > 0$ *depending only on* R *such that for all* $0 < \epsilon < \epsilon_0$*, there exists a finite Blaschke product* $B(z)$ *with all zeros in* E *such that*

$$\frac{1}{K} \sup_{|z| \leq R} |B(z)| \leq C_p(E, \epsilon, R) \leq K \times \sup_{|z| \leq R} |B(z)|^{\alpha\sigma}.$$

These results are in fact corollaries of a more general result (see Corollary 3) which needs only the condition that $M_n(E) \lesssim n^{-\sigma}$ for some constants $\sigma > 0$ and all $n \geq 0$.

Let us remark some features of the set functions $M_n(E)$ in Step 2 above. They are analogous to the set functions defined in (weighted) potential theory for subsets of $\mathbb{C}$ (however there are important differences, see Section 3 for more detailed). In fact in case E is compact in $\mathbb{D}$, we choose $q(z) = 1$, and the function $M_n(E)$ is similar to the classical potential theory for the unit disk (see for example [15]). In a next paper of the second author, it is shown that by choosing a suitable function $q(z)$ these set functions can be defined for all subsets E of $\mathbb{D}$ (not only sets E with $m(E_0) = 0$ as dealt with in this paper), which give a uniform estimate to a quantity analogous to $C_p(E, \epsilon, R)$.

Our approach using interpolation by finite Blaschke products also give a simple and constructive proof to the following result by Danikas[1] and Hayman[2] (see also [4] for a related result)

Theorem 6. *Assume that E is a non-Blaschke sequence (z_j). Then there exists a sequence of positive numbers (η_j) with the property that*

$$\lim_{j\to\infty} \eta_j = 0,$$

such that if f is a non-zero bounded analytic funtion on U then

$$\limsup_{j\to\infty} \frac{|f(z_j)|}{\eta_j} = \infty.$$

This paper is organized as follows. In Section 2 we recall the formula for interpolation by finite Blaschke product, some properties of finite Blaschke product, and give a proof of Theorem 6. In Section 3 we define set functions $M_n(E)$ and other set functions, and the function $\varphi(\epsilon)$ used in Theorem 4. We prove Theorem 4 in Section 4. We prove Corollaries 1 and 2 and give some other examples in Section 5. In Section 6 we prove the similar results for the one-point estimates of $C_p(E, \epsilon, 0)$. In the Appendix we give the proof of (1.5) for the case when $m(E_0) > 0$.

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2. Interpolation by finite Blashcke products

We use the notations in Definition 3.

The following result give an interpolation using Blaschke products for functions in H^p:

Theorem 7. *If $Z_n = (z_1, z_2, \ldots, z_n)$ is a sequence of n distinct points in $\mathbb{D}$ then, for all f in H^p and z in $\mathbb{D}$, the following inequality holds:*

$$\left| f(z) - \sum_{k=1}^{n} c_k(Z_n, z) f(z_k) \right| \leq \frac{\|f\|_p}{(1-|z|^2)^{\frac{1}{p}}} |B(Z_n, z)|, \tag{2.1}$$

where

$$c_{p,k}(Z_n, z) = \frac{1-|z_k|^2}{1-\overline{z_k}z}\left(\frac{1-\overline{z}z_k}{1-|z|^2}\right)^{\frac{2-p}{p}} \frac{B(Z_{n,k}, z)}{B(Z_{n,k}, z_k)}. \tag{2.2}$$

The reader is referred to [5] or [14] for proof of this Theorem.

We need some estimates of $B(Z_n, z)$ and $B(Z_{n,k}, z)$, whose proofs are straightforward.

Proposition 1.

$$|B(Z_n, z)| \leq \exp\left(-\frac{1-|z|^2}{4}\sum_{j=1}^{n}(1-|z_j|)\right),$$

$$|B_k(Z_n, z)| \leq 2\exp\left(-\frac{1-|z|^2}{4}\sum_{j=1}^{n}(1-|z_j|)\right),$$

for z in $\overline{\mathbb{D}}$ and Z_n in $\overline{\mathbb{D}}^n$.

Now we prove Theorem 6

Proof. (of Theorem 6) From properties of E, we can choose a sequence of integers $n_1 < n_2 < ... < n_k < ...$ such that

$$\sum_{j=n_k}^{n_{k+1}-1}(1-|z_j|) \geq k.$$

It follows that $m_k = n_{k+1} - n_k \geq k$. We denote $Z_{(k)} = \{z_{n_k}, z_{n_k+1}, ..., z_{n_{k+1}-1}\}$ (this notation is used only in this proof and just for the sake of simplicity). Then if $n_k \leq j < n_{k+1}$ we define as before $Z_{(k),j} = \{z_{n_k}, z_{n_k+1}, ..., z_{n_{k+1}-1}\}\backslash\{z_j\}$. We define the sequence η_j as follows

$$\eta_j = \frac{|B(Z_{(k),j}, z_j)|}{m(k)},$$

if $n_k \leq j < n_{k+1}$. It is easy to see that $\eta_j \to 0$ as $j \to \infty$.

Now assume that f is a bounded analytic function satisfying

$$\limsup_{j\to\infty}\frac{|f(z_j)|}{\eta_j} < \infty,$$

we will show that $f \equiv 0$. Indeed, fixed $z \in U$ with $|z| \leq 1/2$. Applying Theorem 7 for $Z_{(k)}$ and using Proposition 1 we have

$$\begin{aligned} |f(z)| &\leq C\left(1+\sum_{j=n_k}^{n_{k+1}-1}\frac{|f(z_j)|}{m_k\eta_j}\right)\max_{j=n_k,...,n_{k+1}-1}|B(Z_{(k),j}, z)| \\ &\leq C\exp\{(-k+1)/4\}, \end{aligned}$$

for all k. So $f(z) = 0$ for all $|z| \leq 1/2$. Hence $f \equiv 0$. □

We conclude this section by some more estimates on weighted Blashcke products used later on.

Lemma 1. *If R is a real number in $(0,1)$, then there exists a positive number α depending only on R such that for all r in $[0,1]$, the inequality underneath holds,*

$$\max\{R^\alpha, r^\alpha\} \geq \frac{R+r}{1+Rr}. \tag{2.3}$$

Proof. First, we consider the case $r \leq R$. We have $\max\{R^\alpha, r^\alpha\} = R^\alpha$ and

$$\frac{R+r}{1+Rr} \leq \frac{2R}{1+R^2}.$$

Thus, if this is the case, we must choose α in such a way that

$$0 < \alpha \leq \frac{\ln(2R) - \ln(1+R^2)}{\ln R}.$$

Finally, we consider the case $r > R$. The inequality (2.3) is now equivalent to

$$\frac{r^\alpha - r}{1 - r^{\alpha+1}} \geq R.$$

We will show that the function

$$f(r) = \frac{r^\alpha - r}{1 - r^{\alpha+1}}, \ R \leq r \leq 1$$

attains its absolute minimum at R. We have

$$f'(r) = \frac{r^{2\alpha} - \alpha r^{\alpha+1} + \alpha r^{\alpha-1} - 1}{(1 - r^{\alpha+1})^2}.$$

Define

$$g(r) = r^{2\alpha} - \alpha r^{\alpha+1} + \alpha r^{\alpha-1} - 1, \ R \leq r \leq 1,$$

then

$$g'(r) = 2\alpha r^{2\alpha-1} - \alpha(1+\alpha) r^\alpha - \alpha(1-\alpha) r^{\alpha-2}.$$

By Holder inequality

$$\frac{x^p}{p} + \frac{y^q}{q} \geq xy, \ x, y \geq 0, \ \frac{1}{p} + \frac{1}{q} = 1,$$

applied to

$$\begin{aligned} x &= r^{1-\alpha}, \\ y &= r^{-(1+\alpha)}, \\ p &= \frac{2}{1+\alpha}, \\ q &= \frac{2}{1-\alpha}, \end{aligned}$$

one has

$$(1+\alpha) r^{1-\alpha} + (1-\alpha) r^{-(1+\alpha)} \geq 2$$

if $0 < r < 1$, $0 < \alpha < 1$. This shows that $g'(r) \leq 0$ and thus $g(r) \geq g(1) = 0$. As a consequence, $f(r)$ is non-decreasing, hence when $1 \geq r \geq R$ we have

$$f(r) \geq f(R) = \frac{R^\alpha - R}{1 - R^{\alpha+1}}.$$

Therefore, the proof of the lemma is complete once we can show that for sufficiently small α the inequality $f(R) \geq R$ holds. Indeed, this is equivalent to $R^{\alpha-1} + R^{\alpha+1} \geq 2$. Since $0 < R < 1$, it follows that $R^{-1} + R^1 > 2$. Hence, choosing α small enough leads to the desired result. $\square$

Lemma 2. *Fix $1 > R > 0$. Then there exists a constant $\alpha > 0$ depending only on R such that for all holomorphic function $q(z)$ and $Z_n = \{z_1, \dots, z_n\} \in \mathbb{D}^n$, and $B_q(Z_n, z)$ the weighted Blaschke product with zeros in Z_n we have*

$$\sup_{|z|\leq R} |B_q(Z_n, z)|^\alpha \geq |q(0)|^\alpha \prod_{j=1}^{n} \frac{R + |z_j|}{1 + R|z_j|}.$$

In particular, if $q(z)$ is as in Theorem 4, for any $1 \leq k \leq n$ we have

$$\frac{1}{R}|q(0)|^{-\alpha} \sup_{|z|\leq R} |B_q(Z_n, z)|^\alpha \geq \sup_{|z|\leq R} |B(Z_{n,k}, z)|.$$

Proof. By Jensen's formula (see [13])

$$\sup_{|z|\leq R} |B(Z_n, z)| \geq |q(0)| \prod_{j=1}^{n} \max\{R, |z_j|\}.$$

Choose α as in Lemma 1 we have the conclusion of Lemma 2. □

3. Some set functions

We use notations in Sections 1 and 3. Assume throughout this Section that E is a relative closed subset in $\mathbb{D}$ having infinitely many points, whose non-tangential limit points E_0 has Lebesgue measure zero: $m(E_0) = 0$. Fixed $q(z)$ a function provided by Theorem 2, normalized by $||q||_\infty = 1$. (If E is compact in $\mathbb{D}$ we take $q(z) \equiv 1$).

Let us introduce some definitions.

Definition 8. *Let $Z_n = (z_1, z_2, \dots, z_n) \in \overline{\mathbb{D}}^n$. For all $0 \leq j \leq n$ define $Z_j = \{z_1, \dots, z_j\}$, in particular $Z_0 := \emptyset$. Put*

$$V(Z_n) = \prod_{1\leq j\leq n} |B_q(Z_{j-1}, z_j)|, \tag{3.1}$$

$$\mu(z_1, z_2, \dots, z_n) = \sum_{1\leq j\leq n} \frac{1}{|B_j(Z_n, z_j)|}, \tag{3.2}$$

$$M(z_1, z_2, \dots, z_n) = \sup_{z\in E} |B_q(Z_n, z)|. \tag{3.3}$$

The function $V(Z_n)$ in the above definition can be more explicitly written as

$$V(Z_n) = \prod_{j=1}^{n} |q(z_j)| \prod_{1\leq j<k\leq n} |\frac{z_j - z_k}{1 - \overline{z_j} z_k}|. \tag{3.4}$$

Definition 9. *Let E be a subset of $\overline{\mathbb{D}}$ which contains infinitively many points. Put*

$$V_n(E) = \sup_{Z_n\in E^n} V(Z_n), \tag{3.5}$$

$$\mu_n(E) = \inf_{Z_n\in E^n,\ V(Z_n)=V_n(E)} \mu(Z_n), \tag{3.6}$$

$$M_n(E) = \inf_{Z_n\in E^n,\ V(Z_n)=V_n(E)} M(Z_n). \tag{3.7}$$

The set functions defined above are analog to the set functions defined in (weighted) potential theory for subsets of $\mathbb{C}$ (see for example Section 5.5 in [12]). The sequence $Z_n \in \overline{E}^n$ for which $V_n(E) = V(Z_n)$ are analog to the Fekete points. In case $q(z) \not\equiv 1$, $V_n(E)^{2/n(n-1)}$ is an analog of the n-th diameter. However, for $q(z) \not\equiv 1$, V_n has no analog in the weighted potential theory for $\mathbb{C}$. This is because the function $q(z)$ occurs in $V(Z_n)$ only n times instead of $n(n-1)/2$ times.

Lemma 3. *If $V_n(E) = V(z_1, z_2, \dots, z_n)$ then $z_j \in E$ for all $j = 1, 2, \dots, n$, and $|B_q(Z_{n,j}, z_j)| = \sup_{z\in E} |B_q(Z_{n,j}, z)| = M(Z_{n,j})$.*

Proof. Since E has infinitely many points, we have that $V_n(E) > 0$. Hence since

$$\lim_{z\in E, |z|\to 1} |q(z)| = 0,$$

and since $|B(z)| \leq 1$ for any Blaschke product, it follows that $z_j \in E$ for all j.

From the definition of $V(Z_n)$ we see that

$$0 < V(Z_n) = V(Z_{n,j}) \times |B_q(Z_{n,j}, z_j)|.$$

Since $V(Z_n) = V(E_n)$ it follows that $|B_q(Z_{n,j}, z_j)| = M(Z_{n,j})$. □

Proposition 2. *Let $z_1, z_2, \dots, z_n$ and $\zeta_1, \zeta_2, \dots, \zeta_{n+1}$ be points in $\overline{E}$ such that $V(z_1, z_2, \dots, z_n) = V_n$ and $V(\zeta_1, \zeta_2, \dots, \zeta_{n+1}) = V_{n+1}$, then*

$$\mu(\zeta_1, \zeta_2, \dots, \zeta_{n+1}) M(z_1, z_2, \dots, z_n) \leq (n+1).$$

Proof. If z_0 is the point in $\overline{E}$ such that $|B_q(Z_n, z_0)| = \prod_{1\leq j\leq n} d(z_0, z_j)|q(z)| = M(z_1, z_2, \dots, z_n)$, then $M(z_1, z_2, \dots, z_n)V(z_1, z_2, \dots, z_n) = V(z_0, z_1, z_2, \dots, z_n) \leq V_{n+1}$ (see Lemma 3).

Therefore, for $k = 1, 2, \dots, n+1$, we have

$$\begin{aligned} M(z_1, z_2, \dots, z_n) &\leq \frac{V_{n+1}}{V(z_1, z_2, \dots, z_n)} \leq \frac{V(\zeta_1, \zeta_2, \dots, \zeta_{n+1})}{V(\zeta_1, \dots, \zeta_{k-1}, \zeta_{k+1}, \dots, \zeta_n)} \\ &= |q(\zeta_k)| \prod_{1\leq j\neq k\leq n+1} \left|\frac{\zeta_j - \zeta_k}{1 - \overline{\zeta}_j \zeta_k}\right| \leq 1. \end{aligned}$$

It follows that

$$\mu(\zeta_1, \zeta_2, \dots, \zeta_{n+1}) \leq \frac{(n+1)}{M(z_1, z_2, \dots, z_n)}.$$

This proves the proposition. □

Proposition 3. $\lim_{n\to\infty} V_n^{1/n} = \lim_{n\to\infty} M_n = 0.$

Proof. If E is compact in $\mathbb{D}$ then there exists $1 > r > 0$ such that for all $z \in E$ we have $|z| \leq r$. Hence

$$V_n^{1/n} \leq \left(\frac{2r}{1+r^2}\right)^{(n-1)/2} \to 0$$

as $n \to 0$.

We now consider the case in which $\overline{E} \cap \partial\mathbb{D} \neq \emptyset$.

Fix a number $\delta > 0$. By properties of $q(z)$ (see [2]), it follows that there exist an $r < 1$ such that $|z| < r$ whenever $z \in \overline{E}$ and $q(z) > \delta$. For each n, we rearrange

z_1, z_2, ..., z_n so that: there is a constant k_n for which $|q(z_j)| \leq \delta$ for $1 \leq j \leq k_n$, and $|z_j| \leq r$ for $k_n + 1 \leq j \leq n$. We have

$$\begin{aligned} V_n &= \prod_{1 \leq j < l \leq n} d(z_j, z_l) \prod_{1 \leq j \leq n} |q(z_j)| \\ &\leq \prod_{k_n+1 \leq j < l \leq n} d(z_j, z_l) \prod_{1 \leq j \leq k_n} |q(z_j)| \leq \eta^{(n-k_n)(n-k_n-1)/2} \delta^{k_n}, \end{aligned}$$

where $\eta = \frac{2r}{1+r^2}$. It follows that $V_n^{1/n} \leq \eta^{(n-k_n)(n-k_n-1)/2n} \delta^{k_n/n}$. From this, we see that, if $k_n/n \geq 1/3$, then $V_n^{1/n} \leq \delta^{1/3}$, and if $k_n/n < 1/3$, then $V_n^{1/n} \leq \eta^{n/9}$. Hence

$$\limsup_{n\to\infty} V_n^{1/n} \leq \limsup_{n\to\infty} \max\{\delta^{1/3}, \eta^{n/9}\} = \delta^{1/3}.$$

Since δ can be chosen arbitrarily, we deduce $\lim\limits_{n\to\infty} V_n^{1/n} = 0$.

To prove the second part of Proposition 3, we choose $Z_n = \{z_1, \ldots, z_n\} \in E$ so that $V_n(E) = V(Z_n)$. Noting that $|B_q(Z_n, z)| \leq |B_q(Z_{n,j}, z)|$ and $|B_q(Z_{n,j}, z_j)| \leq |B_q(Z_{j-1}, z_j)|$ for all $j = 1, 2, \ldots, n$, using Lemma 3 we have

$$\begin{aligned} M(Z_n) = M(z_1, z_2, \ldots, z_n) = \sup_{z\in E} |B_q(Z_n, z)| &\leq \left(\prod_{1\leq j\leq n} \sup_{z\in E} |B_q(Z_{n,j}, z)|\right)^{1/n} \\ = \left(\prod_{1\leq j\leq n} |B_q(Z_{n,j}, z_j)|\right)^{1/n} &\leq \left(\prod_{1\leq j\leq n} |B_q(Z_{j-1}, z_j)|\right)^{1/n} = V_n(E)^{1/n}. \end{aligned}$$

Taking supremum on all Z_n with $V(Z_n) = V_n(E)$ we obtain

$$M_n(E) \leq V_n(E)^{1/n}.$$

This leads to the convergence of M_n to 0. □

Now we define the function $\varphi(\epsilon)$ in Theorem 4. Applying Proposition 2, there exists a continuous function $h : [1, \infty) \to (0, \infty)$ such that h is non-increasing, $\lim_{x\to\infty} h(x) = 0$ and $M_n \leq h(n)$for all $n \in \boldsymbol{N}$. We can define such an h as follows: First, define $h(n) = \sup_{k\geq n} M_k$. Then $h(n+1) \leq h(n)$, and by Lemma 3, we see that $\lim_{n\to\infty} h(n) = 0$. Then we extend it appropriately.

We take $\epsilon_0 = \dfrac{h(1)}{2}$. Since $\dfrac{h(x)}{x+1}$ is continuous and strictly decreasing, and $\lim_{x\to\infty} h(x) = 0$, we can define a function $\varphi : (0, \epsilon_0) \to (0, \infty)$ as follows:

$$\varphi(\epsilon) = h(x) \text{ iff } \epsilon = \frac{h(x)}{x+1}. \tag{3.8}$$

We note that φ is non-decreasing and $\lim_{\epsilon\to 0} \varphi(\epsilon) = 0$.

4. Proof of Theorem 4

Fix $R > 0$. Let $g_p(E, \epsilon, R)$ be as in (1.7), let ε_0 and $\varphi(\epsilon)$ be as in the previous section. Let $\alpha > 0$ be the constant in Lemma 1.

Proof. (of Theorem 4) By definition of $C_p(E, \epsilon, R)$ and $g(E, \epsilon, R, q)$, recall that $||q||_\infty = 1$ it follows that $g(E, \epsilon, R, q) \leq C_p(E, \epsilon, R)$. Hence it remains to prove the right hand-sided inequality of (1.8).

Let $Z_n = (z_1, \dots, z_n) \in E^n$. It follows from Theorem 7 that

$$C_p(E, \epsilon, R) \leq K \times \varepsilon\mu(Z_n) \times \max_{1\leq k\leq n} \sup_{|z|\leq R} |B(Z_{n,k}, z)| + K \sup_{|z|\leq R} |B(Z_n, z)|, \tag{4.1}$$

for some constant $K > 0$ depending only on R and p. Applying Lemma 1, remark that $0 < \alpha < 1$, we obtain

$$C_p(E, \epsilon, R) \leq K \times |q(0)|^{-\alpha} \times (\varepsilon\mu(Z_n) + 1) \times \sup_{|z|\leq R} |B_q(Z_n, z)|^{\alpha}. \tag{4.2}$$

It follows from Proposition 3 that $\lim_{n\to\infty} M_n(E) = 0$. Thus, we can choose the smallest n_0 such that $M_{n_0}(E) \leq \varphi(\varepsilon) < M_{n_0-1}(E)$ for all ε less than ε_0. Then, by Proposition 2

$$M_{n_0}(E) \leq \varphi(\varepsilon) < M_{n_0-1}(E) \leq \frac{n_0}{\mu(Z_{n_0})},$$

for any finite sequence $Z_{n_0} = \{z_1, \dots, z_{n_0}\} \in E^{n_0}$ with $V(Z_{n_0}) = V_{n_0}(E)$, $M(Z_{n_0}) = M_{n_0}(E)$. In particular, for such Z_{n_0} we have $\varphi(\varepsilon)\mu(Z_{n_0}) \leq n_0$.

On the other hand, we have $\varphi(\epsilon) < M_{n_0-1}(E) \leq h(n_0 - 1)$ for $n_0 \geq N$. This and (3.8) give $n_0 \leq x + 1$, where

$$\epsilon = \frac{h(x)}{x+1}.$$

Hence,

$$\epsilon\mu(Z_{n_0}) = \frac{\epsilon}{\varphi(\epsilon)}\varphi(\epsilon)\mu(Z_{n_0}) \leq \frac{\epsilon}{\varphi(\epsilon)}n_0 \leq \frac{\epsilon}{\varphi(\epsilon)}(x+1) = \frac{\frac{h(x)}{x+1}}{h(x)}(x+1) = 1. \tag{4.3}$$

Now, $M(Z_{n_0}) = M_{n_0}(E) \leq \varphi(\varepsilon)$ implies that

$$\sup_{|z|\leq R} |B_q(Z_{n_0}, z)| \leq g(E, \varphi(\epsilon), R, q).$$

This, together with (4.3), plugged into (4.2) yields

$$C_p(E, \epsilon, R) \leq 2K \times |q(0)|^{-\alpha} \times g^{\alpha}(E, \varphi(\epsilon), R, q).$$

This concludes the proof of Theorem 4. □

5. Corollaries and examples

We keep the same assumptions as in Section 4.

Corollary 3. *If there exist $C > 0$, $\sigma > 0$ and $N > 0$ such that $M_n(E) \leq Cn^{-\sigma}$ for all $n \geq N$ then there exists $\epsilon_0 > 0$ depending only on E and there exists $\kappa > 0$ depending only on σ such that*

$$g(E, \epsilon, R, q) \leq C_p(E, \epsilon, R) \leq K \times |q(0)|^{-\alpha} \times g^{\alpha}(E, \epsilon^{\sigma}, R, q). \tag{5.1}$$

Proof. If $M_n(E) \leq Cn^{-\sigma}$ for all $n \geq N$ then we choose $h(x) = Cx^{-\sigma}$. So we have

$$\varphi(\epsilon) = h(x) = Cx^{-\sigma} \leq C_2\epsilon^{\sigma/(1+\sigma)},$$

since $\epsilon = Cx^{-\sigma}(1+x)^{-1}$. Applying Theorem 4 completes the proof of Corollary 3. □

Proof. (Of Corollary 1) Since $\overline{E} \subset U$, there exists $0 < r < 1$ such that $\sup_{z\in E} |z| \leq r$. We can also choose $q \equiv 1$. Hence we get that $M_n(E) \leq (\dfrac{2r}{1+r})^n$. So the function $\varphi(\epsilon)$ in Theorem 4 satisfies

$$\lim_{\epsilon\to 0} \frac{\log\epsilon}{\log\varphi(\epsilon)} = 1. \tag{5.2}$$

Applying Theorem 4 completes the proof of Corollary 1. □

Corollary 2 is a consequence of Corollary 3, because of the following result

Proposition 4. *Assume that E is contained in some Stolz angles. Then there exist σ, C and $N > 0$ such that $M_n(E) \leq Cn^{-\sigma}$ for $n \geq N$.*

Proof. Let $\overline{E} \cap \partial U = \{a_1, a_2, ..., a_n\}$. We take in this case $q(z) = (z - a_1)(z - a_2)...(z - a_n)$.

We seperate the proof into three steps.

1. Suppose that $\overline{E}$ lies inside U. In this case $M_n \leq n^{-\sigma}$ for sufficiently large n (see Corollary 1).
2. Suppose that $\overline{E} \cap \partial U$ has only one point. By means of some rotation, we may assume that it this point is 1.

   We have $q(z) = z - 1$. We see that if $|q(z)| > \delta > 0$ for some z in E, then $|z| < r_\delta = 1 - c\delta$ where c is a constant depending on the Stolz angle with vertex at 1. Refering to the proof of Proposition 3, we get
   $$V_n^{1/n} \leq C\max\{\delta^{1/3}, \eta^{n/9}\}. \tag{5.3}$$
   Choosing $\delta = n^{-3\sigma}$ ($\sigma \in (0, 1/6)$), we have
   $$\eta = \frac{2r_\delta}{1 + r_\delta^2} = \frac{2(1 - cn^{-3\sigma})}{1 + (1 - cn^{-3\sigma})^2} = \frac{2n^{6\sigma} - 2cn^{3\sigma}}{2n^{6\sigma} - 2cn^{3\sigma} + c^2}.$$
   Hence,
   $$\eta^{n/9} = \left(\frac{2n^{6\sigma} - 2cn^{3\sigma}}{2n^{6\sigma} - 2cn^{3\sigma} + c^2}\right)^{n/9} = \left(1 - \frac{c^2}{2n^{6\sigma} - 2cn^{3\sigma} + c^2}\right)^{n/9}$$
   $$\leq \left(1 - \frac{c^2}{2n^{6\sigma}}\right)^{\frac{2n^{6\sigma}}{c^2}\frac{c^2n^{1-6\sigma}}{18}} \leq \exp\left(-\frac{c^2n^{1-6\sigma}}{18}\right) \leq n^{-\sigma}$$
   for sufficiently large n. Combining with (5.3), the assertion follows.
3. Now, consider the general case. It suffices to show that if E_1 and E_2 are two sets satisfy $V_n^{1/n}(E_i) \leq Cn^{-\sigma_i}$ for $n \geq N$ ($i = 1, 2$) and $E = E_1 \cup E_2$, then $V_n^{1/n}(E) \leq Cn^{-\sigma}$, for $n \geq 2N$ and $\sigma = \min\{\sigma_1, \sigma_2\}/2$. We take $q(z) = q_1(z)q_2(z)$where q_1, q_2 are coressponding $q's$ functions of E_1, E_2. Fix an $n \geq 2N$ and suppose that $V_n(E) = V(z_1, z_2, \dots, z_l, \zeta_1, \zeta_2, \dots, \zeta_k)$ for $z_j \in E_1$, $\zeta_j \in E_2$, and $n = l + k$. It follows from definitions that
   $$V_n^{1/n}(E) \leq V_l^{1/n}(E_1)V_k^{1/n}(E_2).$$
   We may assume that $l \geq k$. It implies that $l \geq n/2 \geq N$. If $k \leq N$, we have
   $$V_n^{1/n}(E) \leq CV_l^{1/n}(E_1) \leq Cl^{-\sigma_1 l/n} \leq C(n/2)^{-\sigma_1/2} \leq Cn^{-\sigma}.$$

If $k \geq N$, we have

$$V_n^{1/n}(E) \leq V_l^{1/n}(E_1)V_k^{1/n}(E_2) \leq Cl^{-\sigma_1 l/n}k^{-\sigma_2 k/n} \leq C\left(l^{-l/n}k^{-k/n}\right)^{-\sigma}$$
$$= Cn^{-\sigma}\left((l/n)^{-l/n}(k/n)^{-k/n}\right)^{-\sigma} \leq Cn^{-\sigma}.$$

Here we have used the inequality $x^x(1-x)^{1-x} \geq 1/2$ for all $x \in (0,1)$.
The proof is complete. □

We conclude this section providing more sets E satisfying the condition of Corollary 3. For convenience, we recall some definitions that Hayman used in constructing the function f in Theorem 2.

Definition 10. *Let E satisfy (G). We write*

$$E' = \{z = re^{i\theta}:\ |\theta - \phi| < 1 - r \text{ and } re^{i\phi} \in E\}.$$

Next, for $0 \leq \theta \leq 2\pi$, we define

$$\rho(\theta) = \sup\{\rho:\ 0 \leq \rho < 1,\ \rho e^{i\theta} \in E'\}.$$

Let E_∞ be the set of θ such that $\rho(\theta) = 1$. If $\theta \in E_\infty$ then $e^{i\theta} \in E_0$. So $m(E_\infty) = 0$, where $m(.)$ is the Lebesgue's measure of the unit circle.

For each $1 > r > 0$ let E_r be the set of all θ such that $0 \leq \theta \leq 2\pi$ and $\rho(\theta) > r$. Then E_r are open and contract with increasing r, and

$$\bigcap_r E_r = E_\infty.$$

Thus

$$\lim_{r\to 1} m(E_r) = 0.$$

Considering carefully the construction in the proof of Theorem 1 in [2] and Step 2 of the proof of Proposition 4 we can show that if the quantities $m(E_r)$ tend to 0 sufficiently fast, then $M_n \leq Cn^{-\sigma}$. In particular, this claim is true if the following condition is satisfied

$$m(E_\delta) \leq \frac{1}{-2\log\epsilon} \text{ if } \delta = 1 - K\sqrt{-\epsilon\log\epsilon},$$

where K is a positive constant. In fact, if this condition holds, the function f is constructed in Theorem 1 in [2] will satisfy: if $|f(z)| > \epsilon$ then $|z| \leq 1 - K\sqrt{-\epsilon\log\epsilon}$. This last inequality ensures that E satisfies conditions of Corollary 3 (see proof of Proposition 3).

6. One-point estimate

In this section we sketch how to obtain similar results for the case of one-point estimate, that is of estimating $C_p(E,\epsilon,0)$ in (1.4). There are two cases:

Case 1: $0 \in E$. In this case it is easy to see that $C_p(E,\epsilon,0) = \epsilon$.

Case 2: $0 \notin E$. In this case there exists $1 > r > 0$ such that if $z \in E$ then $|z| \geq r$. Then we can define similar set functions like those in Section 3 to obtain similar result to that of Theorem 4 and Corollaries 1, 2 and 3.

7. Appendix: Case $m(E_0) > 0$

In this section we present the proof of (1.5) when $m(E_0) > 0$. We thank Professor Yuril Lyubarskii for showing us this proof.

Proof. (Of (1.5))

Since $m(E_0) > 0$ the harmonic measure $\omega(z)$ of E_0 (see [12]) satisfies: ω is a harmonic function in $\mathbb{D}$, $0 < \omega(z) < 1$ for all $z \in \mathbb{D}$, (its boundary value) $\omega(z) = 1$ a.e for $z \in E_0$, $\omega(z) = 0$ for a.e $z \in \partial\mathbb{D}\backslash E_0$. Let $v(z)$ be an analytic function with real part ω.

For any $\varepsilon > 0$ define

$$u_\varepsilon(z) = \exp\{\log\varepsilon \times v(z)\} = \varepsilon^{v(z)}.$$

Then u_ε is analytic in $\mathbb{D}$, $0 < |u_\varepsilon(z)| = \varepsilon^{\omega(z)} < 1$ for all $z \in \mathbb{D}$, $|u_\varepsilon(z)| = \varepsilon$ a.e for $z \in E_0$, $|u_\varepsilon(z)| = 1$ for a.e $z \in \partial\mathbb{D}\backslash E_0$.

Let f be any function in $\mathcal{A}^p$ with $|f(z)| \leq \varepsilon$ for all $z \in E$. Then $|f(z)| \leq \varepsilon$ a.e in E_0. Then f/u_ε is holomorphic in $\mathbb{D}$ and we have

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} |\frac{f}{u_\varepsilon}(e^{it})|^p dt &= \frac{1}{2\pi}\int_{t\in E_0} \frac{|f(e^{it})|^p}{|\varepsilon|^p}dt + \frac{1}{2\pi}\int_{t\not\in E_0} \frac{|f(e^{it})|^p}{1}dt \\ &\leq \frac{1}{2\pi}\int_{t\in E_0} dt + \frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^p dt \leq 2. \end{aligned}$$

Hence $||f/u_\varepsilon||_{H^p} \leq 2^{1/p}$. Applying (1.2) to f/u_ε and use the definition of u_ε we obtain (1.5). □

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